CHAPTER 8

Three-Dimensional Small-Disturbance Solutions

Chapter 4:

1. The small-disturbance problem for a wing was established.
2. The problem is separated into the solution of two linear sub-problems, namely the thickness and lifting problems.

Chapter 5:

The thickness and lifting problems for airfoil was solved. These solutions were added to yield the complete small-disturbance solution for the flow past a thin airfoil.

Chapter 8:

In this chapter, 3D small-disturbance solutions will be derived for some simple cases such as the large aspect ratio wing, the slender pointed wing, and the slender cylindrical body.

8.1 Finite Wing: The Lifting Line Model

Formolating 3D lifting wing problem (Chapter 4) → *Difficulty of obtaining an analytic solution from the integral equations*

↓

Approximating the lifting properties of a wing by a single lifting line (closed-form solution)

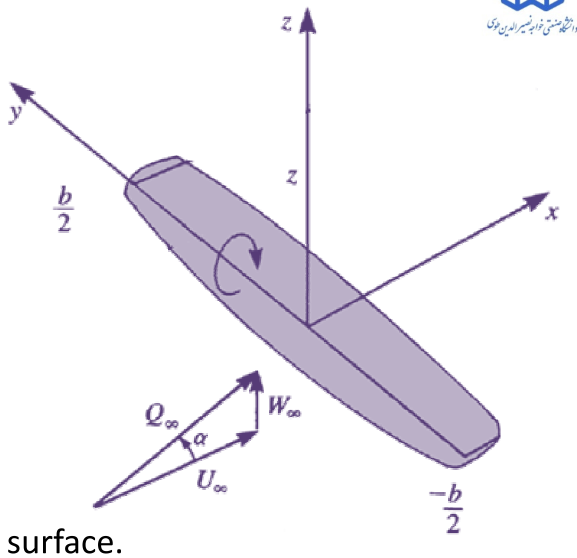
↓ *considerable simplifications*

Capturing the basic features of 3D lifting flows →

I. Increasing in induced drag with decreasing aspect ratio

II. Reduction of lift slope

8.1 Finite Wing: The Lifting Line Model



Definition of the Problem

- Lifting, thin, finite wing
- Moving at a constant speed
- Small angle of attack

Laplace's equation for the perturbation potential:

$$\nabla^2 \Phi = 0 \qquad (8.1)$$

B.C. On wing solid surface approximated at z = 0:

$$\frac{\partial \Phi}{\partial z}(x, y, 0\pm) = Q_\infty \left(\frac{\partial \eta}{\partial x} - \alpha \right) \qquad (8.2)$$

Selecting a vortex distribution for modeling the lifting surface.
The unknown vortex distribution γ_x (x, y) and γ_y (x, y) is placed on the wing's projected area at the z = 0 plane.

$$\frac{-1}{4\pi} \int_{\text{wing+wake}} \frac{\gamma_y(x - x_0) - \gamma_x(y - y_0)}{[(x - x_0)^2 + (y - y_0)^2]^{3/2}} \, dx_0 \, dy_0 = Q_\infty \left(\frac{\partial \eta}{\partial x} - \alpha \right) \qquad (8.3)$$

Satisfying Kutta condition at T.E. for a proper & unique solution for vortex distribution:

$$\gamma_{\text{T.E.}} = 0$$

Vortex lines do not begin or end in a fluid: ➡ $\left| \frac{\partial \gamma_x}{\partial x} \right| = \left| \frac{\partial \gamma_y}{\partial y} \right|$

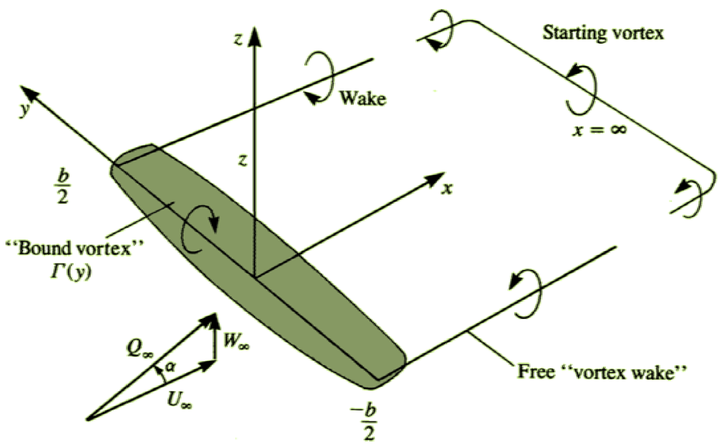
8.1 Finite Wing: The Lifting Line Model

The Lifting-Line Model

- The chordwise circulation, at any spanwise station, is replaced by a single concentrated vortex.
- These local vortices of circulation Γ(y) will be placed along a single spanwise line.
- This vortex line will be placed at the wing's quarter-chord (c/4) line along the span, −b/2 < y < b/2 (Based on results of section 5.5 for 2D lumped-vortex element).
- Trailing vortices must be shed into the flow to create a wake such that there will be no force acting on these free vortices.

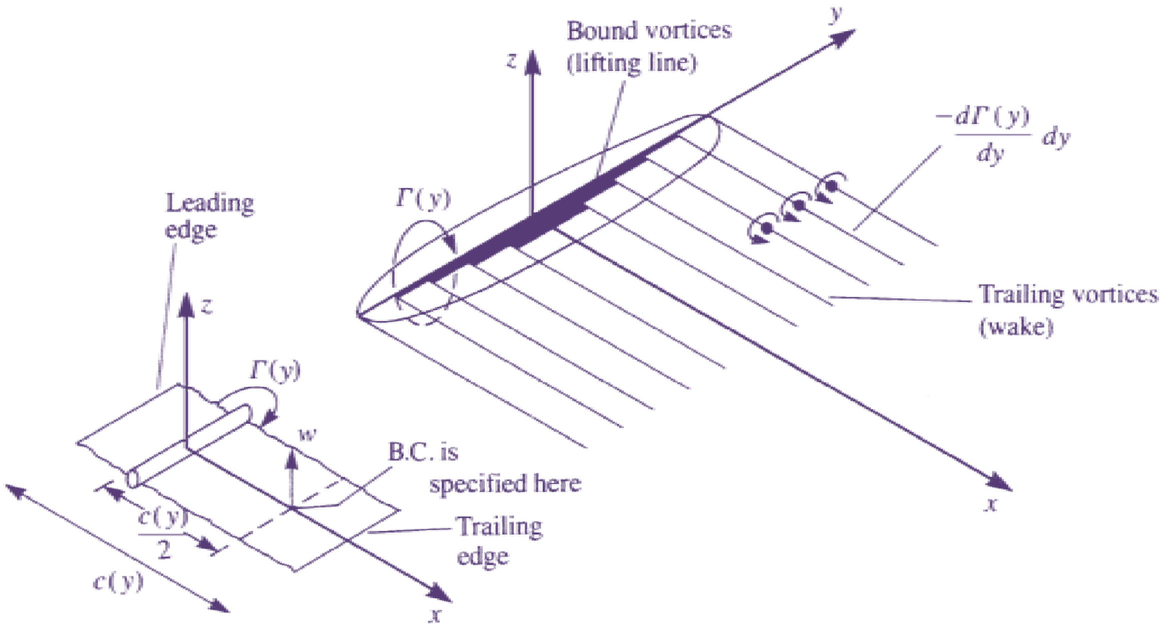
$$\mathbf{q} \times \mathbf{\Gamma}_{\text{wake}} = 0$$

The most basic element that will fit these requirements will have the shape of a horseshoe vortex , which will have constant bound vorticity Γ along its c/4 line, will turn backward at the wing tips and will continue far behind the wing, and eventually will be closed by the starting vortex. It is assumed here that the flow is steady and therefore the starting vortex is far downstream and its influence can be neglected.



8.1 Finite Wing: The Lifting Line Model

A more refined model of the finite wing was first proposed by the German scientist Ludwig Prandtl during World War 1 and it uses a large number of such spanwise horseshoe vortices.



The straight bound vortex Γ(y) in this case is placed along the y axis and at each spanwise station, L.E. is one-quarter chord ahead of this line.

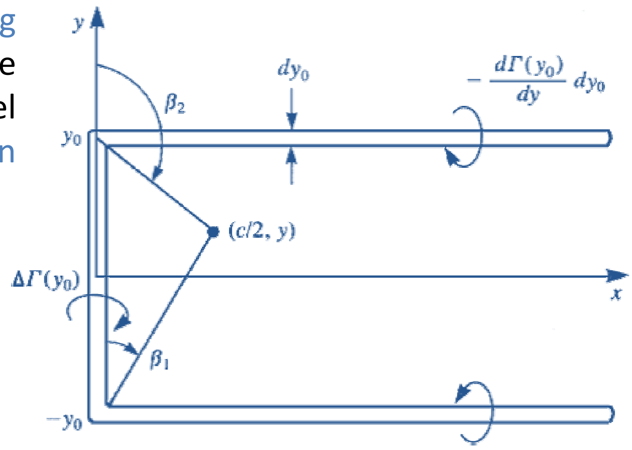
8.1 Finite Wing: The Lifting Line Model

For the case of the flat lifting surface, where $\partial\eta/\partial x = 0$. The equation now simply states B.C. of Eq. (8.2):

$$\frac{\partial \Phi_{\text{wing}}}{\partial z} + \frac{\partial \Phi_{\text{wake}}}{\partial z} + Q_\infty \alpha = 0 \qquad \textbf{OR} \qquad w_b + w_i + Q_\infty \alpha = 0 \qquad (8.7)$$

The normal velocity components induced by the wing (w_b) & wake vortices (w_i).

The velocity component w_b induced by the lifting line on the section with a chord c(y) can be estimated by using the lumped-vortex model with the downwash calculated at the collocation point located at the 3/4 chord.



A typical horseshoe vortex with strength $\Delta\Gamma(y_0) = -(d\Gamma(y_0) / dy)\, dy_0$

$$\Delta w_b = \frac{-\Delta\Gamma}{4\pi d}(\cos\beta_1 - \cos\beta_2)$$

$$= \frac{-\Delta\Gamma}{4\pi[c(y)/2]}\left[\frac{y+y_0}{\sqrt{(c/2)^2+(y+y_0)^2}} + \frac{y_0-y}{\sqrt{(c/2)^2+(y_0-y)^2}}\right]$$

8.1 Finite Wing: The Lifting Line Model

For a wing of large aspect ratio, we can neglect $(c/2)^2$ in the square root terms to get

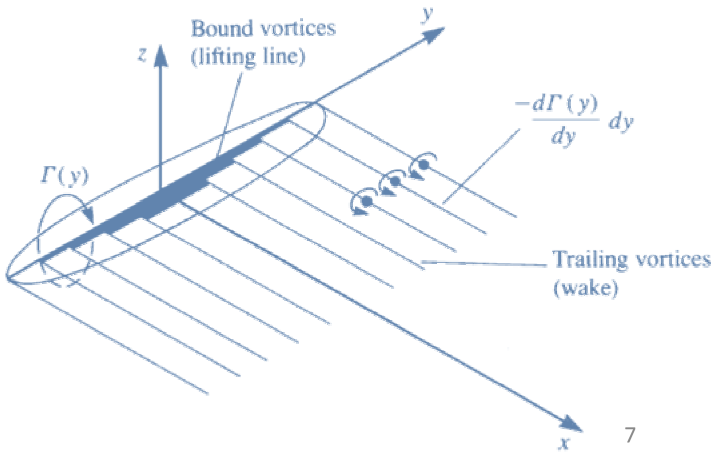
$$\Delta w_b = \frac{-\Delta\Gamma(y_0)}{4\pi\,[c(y)/2]}[1+1]$$

The result for the complete lifting line (evaluated at y) is obtained by summing the results for all the horseshoe vortices

$$w_b = \frac{-\Gamma(y)}{2\pi\,[c(y)/2]} \qquad (8.8)$$

Note: this is identical to the result obtained by applying a locally two-dimensional lumped-vortex model at each spanwise station

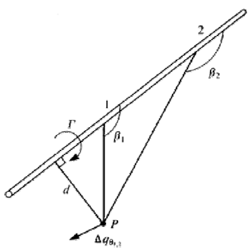
The wake is constructed from semi-infinite vortex lines with the strength of $(d\Gamma/dy)dy$



8.1 Finite Wing: The Lifting Line Model

The right-hand-side wake vortex line is located at a spanwise location y_0 and the downwash induced by this vortex at the collocation point (c/2, y) is given by the result for a semi-infinite vortex line from Eq. (2.71).

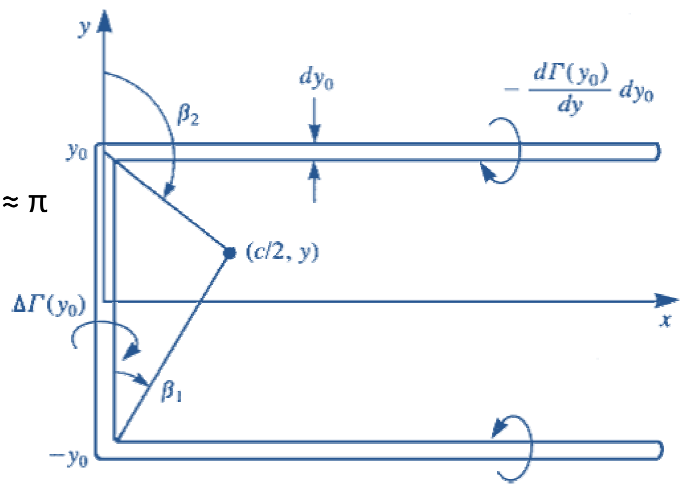
$$q_\theta = \frac{\Gamma}{4\pi d} \qquad (2.71)$$



Since for a large aspect ratio wing $\beta_1 \approx \pi/2$, $\beta_2 \approx \pi$

$$w(y) = \frac{\Delta\Gamma(y_0)}{4\pi}\frac{1}{(y-y_0)} \qquad (8.9)$$

Exactly one half of the velocity induced by an infinite (2D) vortex of strength $\Delta\Gamma(y_0)$



The normal velocity component induced by the trailing vortices of the wing becomes

$$w_i = \frac{1}{4\pi}\int_{-b/2}^{b/2}\frac{[-d\Gamma(y_0)/dy]\,dy_0}{y-y_0} \qquad (8.10)$$

8.1 Finite Wing: The Lifting Line Model

Assuming that the wing aspect ratio is large (b/c(y)>>1) has allowed us to treat a spanwise station as a 2D section & to transfer the B.C. to the local 3/4 chord.

$$w_b = \frac{-\Gamma(y)}{2\pi[c(y)/2]} \qquad w_i = \frac{1}{4\pi}\int_{-b/2}^{b/2} \frac{[-d\Gamma(y_0)/dy]\,dy_0}{y-y_0}$$

$$w_b + w_i + Q_\infty \alpha = 0$$

$$\frac{-\Gamma(y)}{2\pi[c(y)/2]} - \frac{1}{4\pi}\int_{-b/2}^{b/2} \frac{[d\Gamma(y_0)/dy]\,dy_0}{y-y_0} + Q_\infty \alpha = 0 \qquad (8.11)$$

Dividing by Q_∞

$$\frac{-\Gamma(y)}{\pi c(y) Q_\infty} - \frac{1}{4\pi Q_\infty}\int_{-b/2}^{b/2} \frac{[d\Gamma(y_0)/dy]\,dy_0}{y-y_0} + \alpha = 0 \qquad (8.11a)$$

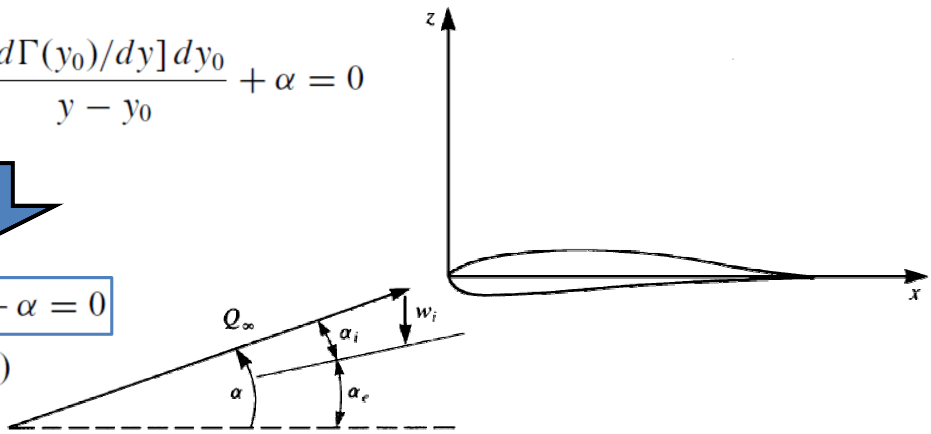
The Prandtl lifting-line integrodifferential equation for the spanwise load distribution Γ(y)

8.1 Finite Wing: The Lifting Line Model

$$\frac{-\Gamma(y)}{\pi c(y) Q_\infty} - \frac{1}{4\pi Q_\infty}\int_{-b/2}^{b/2} \frac{[d\Gamma(y_0)/dy]\,dy_0}{y-y_0} + \alpha = 0$$

can be viewed as a combination of the angles

$$\boxed{-\alpha_e - \alpha_i + \alpha = 0} \qquad (8.12)$$



Induced downwash angle is (note that w is positive in the positive z direction):

$$\alpha_i \approx \frac{-w_i}{Q_\infty} \qquad (8.13)$$

Rearranging Eq. (8.12):

$$\alpha_e = \alpha - \alpha_i \qquad (8.12a)$$

In the case of the finite wing the effective AOA of a wing section α_e is smaller than the actual geometric AOA α by α_i, which is a result of the downwash induced by the wake.

8.1 Finite Wing: The Lifting Line Model

It is possible to generalize the result of this equation by assuming that the 2D section has a local lift slope of m_0 and its local effective AOA is α_e. Now, if camber effects are to be accounted for too, then this angle is measured from the zero-lift angle of the section, such that:

$$C_l(y) = \frac{\rho Q_\infty \Gamma(y)}{(1/2)\rho Q_\infty^2 c(y)} = m_0(y)\alpha_e(y) \qquad (8.14)$$

Eq. (8.12a) becomes

$$\alpha_e = \alpha - \alpha_i - \alpha_{L0} \qquad (8.15)$$

α_{L0} is the angle of zero lift due to the section

A more general form of Eq. (8.11a) that allows for section camber & wing twist $\alpha(y)$ is:

$$\frac{-2\Gamma(y)}{m_0(y)c(y)Q_\infty} - \frac{1}{4\pi Q_\infty}\int_{-b/2}^{b/2}\frac{[d\Gamma(y_0)/dy]\,dy_0}{y-y_0} + \alpha(y) - \alpha_{L0}(y) = 0 \qquad (8.16)$$

- *local angle of attack relative to $\mathbf{Q_\infty}$*
- *$\alpha_{L0}(y)$ is the airfoil section zero-lift angle*

} *Known Geometrical quantities* ➜ *Γ(y) is unknown*

\+

wing tips pressure difference or lift = zero ➜

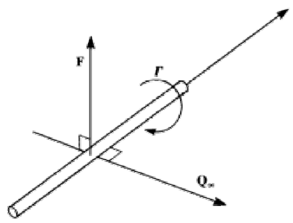
$$\Gamma\left(y = \pm\frac{b}{2}\right) = 0 \qquad (8.17)$$

8.1 Finite Wing: The Aerodynamic Loads

Solution of Eq. (8.16) ➜ Spanwise bound circulation distribution Γ(y)

To obtain the aerodynamic forces, the 2D Kutta–Joukowski theorem will be applied (in the y = const. plane).
Because of the wake-induced velocity, the free-stream vector will be rotated by $\alpha_i(y)$
This angle can be calculated for a known Γ(y) by using Eqs. (8.10) and (8.13):

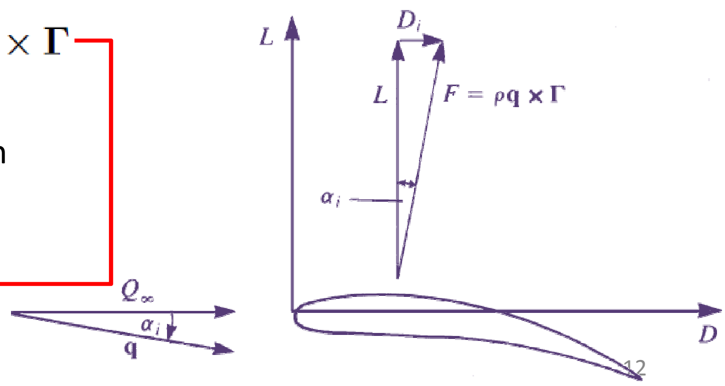
$$\alpha_i = \frac{1}{4\pi Q_\infty}\int_{-b/2}^{b/2}\frac{[d\Gamma(y_0)/dy]\,dy_0}{y-y_0} \qquad (8.18)$$



α_i is small $\begin{cases} \cos\alpha_i \approx 1 \\ \sin\alpha_i \approx \alpha_i \end{cases}$ ➜ $\mathbf{F} = \rho \mathbf{Q}_\infty \times \mathbf{\Gamma}$

The lift of the wing is given by an integration of the local two-dimensional lift (Eq. 3.113)

$$L = \rho Q_\infty \int_{-b/2}^{b/2}\Gamma(y)\,dy \qquad (8.19)$$



8.1 Finite Wing: The Aerodynamic Loads

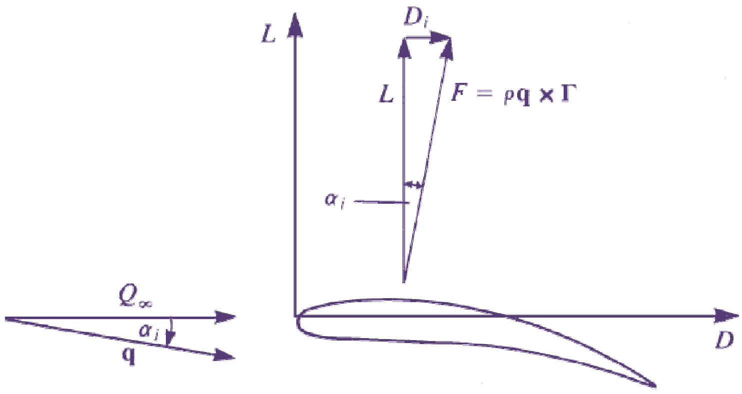
The drag force, which is created by turning the 2D lift vector by the wake-induced flow

$$D_i = \rho Q_\infty \int_{-b/2}^{b/2} \alpha_i(y)\Gamma(y)\,dy \qquad (8.20)$$

induced drag

⇓

induced by the trailing vortices



Eq. (8.20) can be rewritten in terms of the wake-induced downwash w_i

$$D_i = -\rho \int_{-b/2}^{b/2} w_i(y)\Gamma(y)\,dy \qquad (8.20a)$$

The total drag D of a wing includes the induced drag D_i and the viscous drag D_0

$$D = D_i + D_0$$

8.1 Finite Wing: The Elliptic Lift Distribution

The spanwise circulation distribution Γ(y) for a given planform shape can be obtained by solving Eq. (8.16)

$$\frac{-2\Gamma(y)}{m_0(y)c(y)Q_\infty} - \frac{1}{4\pi Q_\infty}\int_{-b/2}^{b/2} \frac{[d\Gamma(y_0)/dy]\,dy_0}{y - y_0} + \alpha(y) - \alpha_{L0}(y) = 0 \qquad (8.16)$$

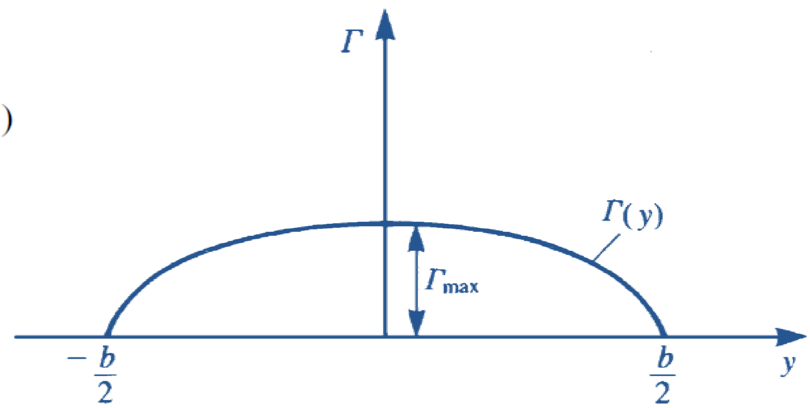
Elliptic distribution of the circulation → downwash w_i becomes constant along the wing span → minimum induced drag

The proposed distribution of Γ(y):

$$\Gamma(y) = \Gamma_{\max}\left[1 - \left(\frac{y}{b/2}\right)^2\right]^{1/2} \qquad (8.21)$$

⇓

substituting into Eq. (8.16)
the constant Γ_{max} can be evaluated



8.1 Finite Wing: The Elliptic Lift Distribution

Eq. (8.21) *differentiating* →

$$\frac{d\Gamma(y)}{dy} = \frac{\Gamma_{max}}{2}\left[1-\left(\frac{y}{b/2}\right)^2\right]^{-1/2}\left(-2\frac{4}{b^2}y\right)$$

Substituting into Eq. (8.10)

$$w_i = \frac{1}{4\pi}\int_{-b/2}^{b/2}\frac{[-d\Gamma(y_0)/dy]\,dy_0}{y-y_0} \qquad (8.10)$$

downwash

$$w_i(y) = \frac{\Gamma_{max}}{\pi b^2}\int_{-b/2}^{b/2}\left[1-\left(\frac{y_0}{b/2}\right)^2\right]^{-1/2}\frac{y_0}{y-y_0}\,dy_0 \qquad (8.22)$$

Note that when $y = y_0$, this integral is singular and therefore must be evaluated based on Cauchy's principal value

8.1 Finite Wing: The Elliptic Lift Distribution

Using Glauert's integral (Eq. (5.22)) by the transformation

$$y = \frac{b}{2}\cos\theta \qquad (8.23)$$

$$dy = -\frac{b}{2}\sin\theta\,d\theta \qquad (8.23a)$$

→ *at the wingtips* $\begin{cases} y = -b/2,\ \theta = \pi \\ y = b/2,\ \theta = 0 \end{cases}$

Eq. (8.21) reduce to

$$\Gamma(\theta) = \Gamma_{max}[1-\cos^2\theta]^{1/2} = \Gamma_{max}\sin\theta \qquad (8.21a)$$

Substituting Eq. (8.23) into Eq. (8.22) →

$$w_i = \frac{\Gamma_{max}}{\pi b^2}\int_{\pi}^{0}[1-\cos^2\theta_0]^{-1/2}\frac{(b/2)\cos\theta_0[(-b/2)\sin\theta_0]d\theta_0}{(b/2)(\cos\theta-\cos\theta_0)}$$

using the Glauert integral →

$$w_i = \frac{-\Gamma_{max}}{2\pi b}\int_0^{\pi}\frac{\cos\theta_0\,d\theta_0}{(\cos\theta_0-\cos\theta)} = \frac{-\Gamma_{max}}{2\pi b}\frac{\pi\sin\theta}{\sin\theta}$$

In elliptic distribution of the circulation, the downwash w_i becomes constant along the wing span

$$w_i = -\frac{\Gamma_{max}}{2b} \qquad (8.24)$$

$\alpha_i \approx \frac{-w_i}{Q_\infty}$ →

$$\alpha_i = \frac{\Gamma_{max}}{2bQ_\infty} \qquad (8.24a)$$

8.1 Finite Wing: The Elliptic Lift Distribution

Another feature of the elliptic distribution is that the spanwise integral is simply half the area of an ellipse (with semi-axes Γ_{max} and b/2)

$$\int_{-b/2}^{b/2} \Gamma(y)\,dy = \frac{\pi}{2}\Gamma_{\max}\frac{b}{2} = \frac{\pi b}{4}\Gamma_{\max} \qquad (8.25)$$

Consequently, the lift and the drag of the wing can be evaluated as:

$$L = \rho Q_\infty \int_{-b/2}^{b/2} \Gamma(y)\,dy = \frac{\pi b}{4}\rho Q_\infty \Gamma_{\max} \qquad (8.26)$$

$$D_i = \rho Q_\infty \int_{-b/2}^{b/2} \alpha_i \Gamma(y)\,dy = \alpha_i L = \left(\frac{\Gamma_{\max}}{2bQ_\infty}\right)\frac{\pi b}{4}\rho Q_\infty \Gamma_{\max} = \frac{\pi}{8}\rho\Gamma_{\max}^2 \qquad (8.27)$$

$$C_L \equiv \frac{L}{(1/2)\rho Q_\infty^2 S} = \frac{\pi}{2}\frac{b}{S}\frac{\Gamma_{\max}}{Q_\infty} \qquad (8.28)$$

$$C_{D_i} \equiv \frac{D_i}{(1/2)\rho Q_\infty^2 S} = \frac{\pi}{4S}\frac{\Gamma_{\max}^2}{Q_\infty^2} = \frac{1}{\pi}\frac{S}{b^2}C_L^2 \qquad (8.29)$$

8.1 Finite Wing: The Elliptic Lift Distribution

Substituting the spanwise downwash, Eq. (8.24) & the elliptic circulation distribution, Eq. (8.21) into Eq. (8.16):

$$\frac{-2\Gamma_{\max}}{m_0(y)c(y)Q_\infty}\left[1-\left(\frac{y}{b/2}\right)^2\right]^{1/2} - \frac{\Gamma_{\max}}{2bQ_\infty} + \alpha(y) - \alpha_{L0}(y) = 0 \qquad (8.30)$$

the relation between the local chord c(y) & the local AOA α(y) for the wing with the elliptic circulation distribution

If the chord c(y) has an elliptic form such as

$$c(y) = c_0\left[1-\left(\frac{y}{b/2}\right)^2\right]^{1/2} \qquad (8.31)$$

root chord

Substituting Eq. (8.31) into Eq. (8.30)

$$\frac{-2\Gamma_{\max}}{m_0(y)c_0Q_\infty} - \frac{\Gamma_{\max}}{2bQ_\infty} + \alpha(y) - \alpha_{L0}(y) = 0 \qquad (8.32)$$

varied for diffrent airfoil shape palnform

varied for twisted wing

8.1 Finite Wing: The Elliptic Lift Distribution

For an elliptic planform with constant airfoil shape, all terms but α(y) in this equation are constant, and therefore this wing with an elliptic planformand load distribution is untwisted (α(y) = α = const.). The value of Γ_{max} is then:

$$\Gamma_{\max} = \frac{2bQ_\infty(\alpha - \alpha_{L0})}{1 + 4b/m_0 c_0} \qquad (8.33)$$

area of elliptic wing $S = \pi \frac{c_0}{2}\frac{b}{2}$

wing aspect ratio $\text{Æ} \equiv \frac{b^2}{S}$

$$\Gamma_{\max} = \frac{2bQ_\infty(\alpha - \alpha_{L0})}{1 + \pi \text{Æ}/m_0} \qquad (8.33a)$$

using $m_0 = 2\pi$

$$C_L = \frac{2\pi}{1 + 2/\text{Æ}}(\alpha - \alpha_{L0}) \equiv C_{L_\alpha}(\alpha - \alpha_{L0}) \qquad (8.36)$$

3D wing lift slope

8.1 Finite Wing: The Elliptic Lift Distribution

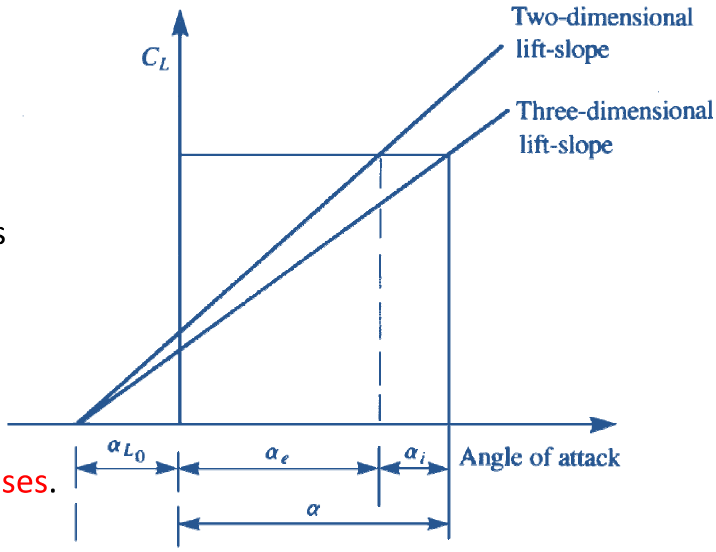
The 3D wing lift slope becomes less as the wing span becomes smaller due to the induced downwash.

$$C_L = \frac{2\pi}{1 + 2/\text{Æ}}(\alpha - \alpha_{L0}) \equiv C_{L_\alpha}(\alpha - \alpha_{L0})$$

for a wing with given α_{L0} the effective AOA is reduced by α_i

$$\alpha_e = \alpha - \alpha_i - \alpha_{L0}$$

For finite span wings, more incidence is needed to achieve the same C_L as AR decreases.



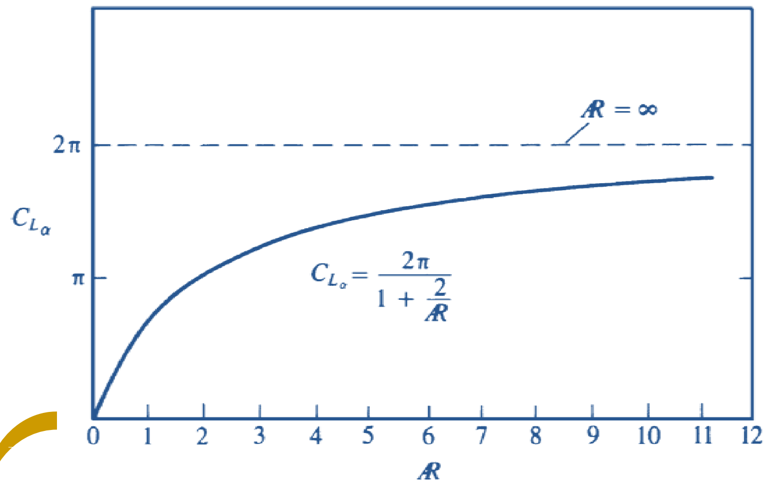
The induced drag coefficient is obtained by substituting Eq. (8.35) into Eq. (8.29):

$$C_{D_i} = \frac{1}{\pi \text{Æ}} C_L^2 \qquad (8.37)$$

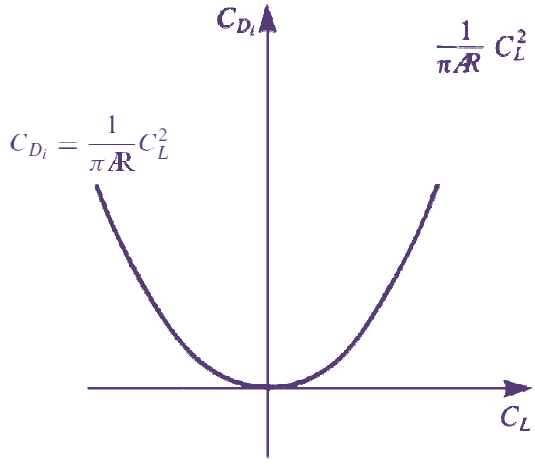
As AR increases the induced drag becomes smaller.

8.1 Finite Wing: The Elliptic Lift Distribution

The induced drag for the finite elliptic wing will increase with a rate of C_L^2



Variation of lift coefficient slope versus aspect ratio for thin elliptic wings Eq. (8.36)



Lift polar for an elliptic wing

The lift slope of a 2D wing is the largest (2π) and as the wing span becomes smaller $C_{L\alpha}$ decreases too.

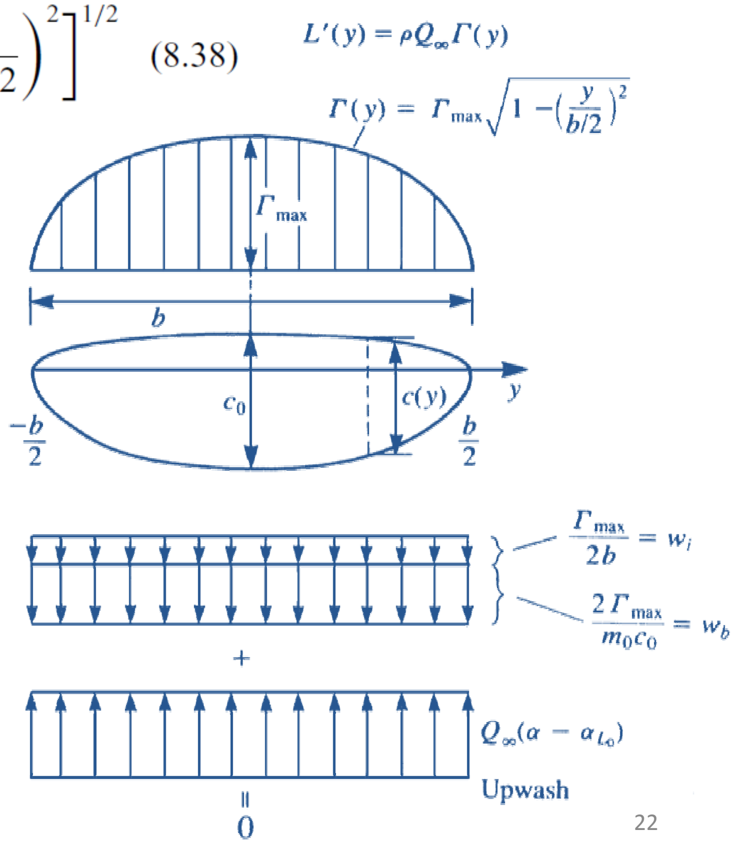
$$C_L = \frac{2\pi}{1 + 2/\mathcal{R}}(\alpha - \alpha_{L0}) \equiv C_{L_\alpha}(\alpha - \alpha_{L0})$$

8.1 Finite Wing: The Elliptic Lift Distribution

The spanwise loading L'(y) (lift per unit span) of the elliptic wing is obtained by using the Kutta–Joukowski theorem:

$$L'(y) = \rho Q_\infty \Gamma(y) = \rho Q_\infty \Gamma_{max}\left[1 - \left(\frac{y}{b/2}\right)^2\right]^{1/2} \quad (8.38)$$

Chord and load distribution for a thin elliptic wing. Note that the induced downwash (w_i) is constant and combined with the downwash of the bound vortex (w_b) is equal to the free-stream upwash ($Q_\infty \alpha$), resulting in zero velocity normal to the wing surface.



Note: that it is possible to have elliptic loading with other than an elliptic planform, but in that case, local twist or camber needs to be adjusted so that w_i will remain constant.

8.1 Finite Wing: The Elliptic Lift Distribution

The section lift coefficient C_l is defined by using the local chord from Eq. (8.31):

$$C_l \equiv \frac{L'(y)}{(1/2)\rho Q_\infty^2 c(y)} = \frac{2\Gamma_{\max}}{c_0 Q_\infty} = C_L$$

Thus, for elliptic wing, both section lift coefficient & wing lift coefficient are the same.

$$C_l = \frac{2\pi}{1 + 2/\mathcal{R}}(\alpha - \alpha_{L0}) \equiv C_{l_\alpha}(\alpha - \alpha_{L0}) \qquad (8.39)$$

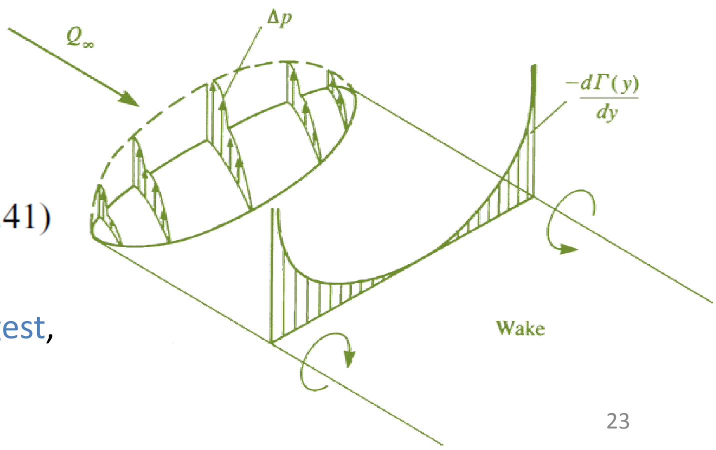
Similarly, the section induced drag coefficient is:

$$C_{d_i} \equiv \frac{L'(y)\alpha_i}{(1/2)\rho Q_\infty^2 c(y)} = \frac{1}{\pi}\frac{S}{b^2}C_l^2 = C_{D_i} \qquad (8.40)$$

The strength of the circulation in the wake is simply the spanwise derivative of Γ(y):

$$\frac{d\Gamma(y)}{dy} = -\frac{4\Gamma_{\max}}{b^2}\frac{y}{\sqrt{[1-(y/(b/2))^2]}} \qquad (8.41)$$

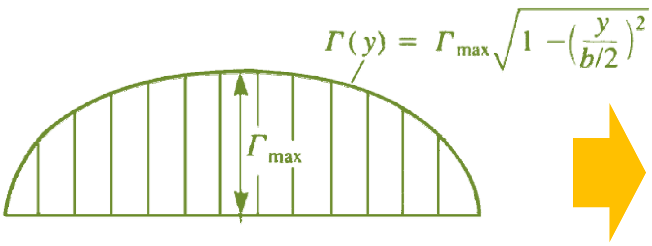
near the wingtips, where |Γ(y)/dy| is the largest, the wake vortex will be the strongest.

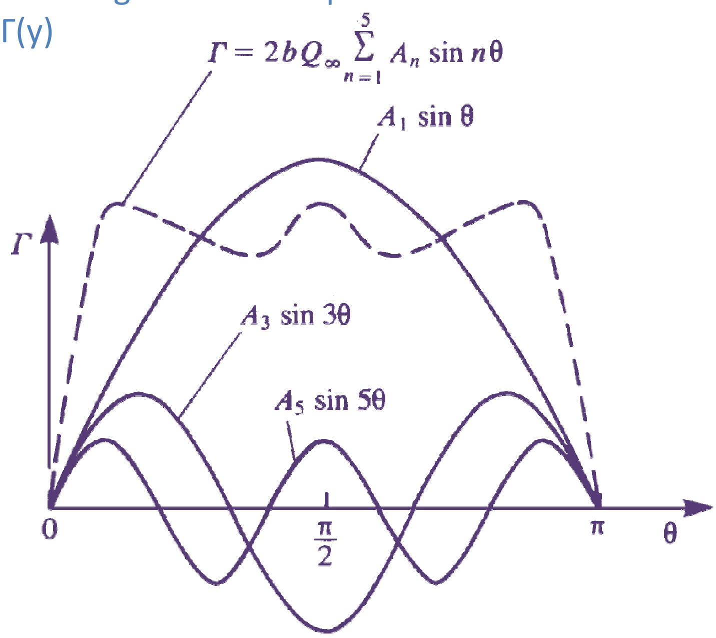


8.1 Finite Wing: General Spanwise Circulation Distribution

Describing the unknown distribution in terms of a trigonometric expansion for more general solution for the spanwise circulation Γ(y)

$$\Gamma(\theta) = 2bQ_\infty \sum_{n=1}^{\infty} A_n \sin n\theta \qquad (8.42)$$





Sine series representation of symmetric spanwise circulation distribution Γ(θ), n =1, 3, 5.

All terms of Fourier expansion fulfill Eq. (8.17) at the wingtips:

$$\Gamma(0) = \Gamma(\pi) = 0 \qquad (8.43)$$

8.1 Finite Wing: General Spanwise Circulation Distribution

Substituting Γ(θ) and dΓ(θ)/dy into Eq. (8.16)

$$\frac{-4b}{m_0(\theta)c(\theta)}\sum_{n=1}^{\infty}A_n\sin n\theta$$

$$+\frac{-b}{2\pi}\int_{\pi}^{0}\frac{\sum_{n=1}^{\infty}A_n n\cos n\theta_0[1/(-b/2)\sin\theta_0][(-b/2)\sin\theta_0 d\theta_0]}{(b/2)(\cos\theta-\cos\theta_0)}$$

$$+\alpha(\theta)-\alpha_{L0}(\theta)=\frac{-4b}{m_0(\theta)c(\theta)}\sum_{n=1}^{\infty}A_n\sin n\theta-\frac{1}{\pi}\int_0^{\pi}\frac{\sum_{n=1}^{\infty}nA_n\cos n\theta_0\,d\theta_0}{\cos\theta_0-\cos\theta}$$

$$+\alpha(\theta)-\alpha_{L0}(\theta)=0 \tag{8.44}$$

Using Glauert's integral for the second term

$$\frac{-4b}{m_0(\theta)c(\theta)}\sum_{n=1}^{\infty}A_n\sin n\theta-\sum_{n=1}^{\infty}nA_n\frac{\sin n\theta}{\sin\theta}+\alpha(\theta)-\alpha_{L0}(\theta)=0 \tag{8.44a}$$

8.1 Finite Wing: General Spanwise Circulation Distribution

$$\frac{-2\Gamma(y)}{m_0(y)c(y)Q_\infty}-\frac{1}{4\pi Q_\infty}\int_{-b/2}^{b/2}\frac{[d\Gamma(y_0)/dy]\,dy_0}{y-y_0}+\alpha(y)-\alpha_{L0}(y)=0 \tag{8.16}$$

$$\underbrace{\frac{-4b}{m_0(\theta)c(\theta)}\sum_{n=1}^{\infty}A_n\sin n\theta}_{-\alpha_e}\underbrace{-\sum_{n=1}^{\infty}nA_n\frac{\sin n\theta}{\sin\theta}}_{-\alpha_i}+\alpha(\theta)-\alpha_{L0}(\theta)=0 \tag{8.44a}$$

Comparing with Eq. (8.16)

$$\alpha_i(\theta)=\sum_{n=1}^{\infty}nA_n\frac{\sin n\theta}{\sin\theta} \tag{8.45}$$

8.1 Finite Wing: General Spanwise Circulation Distribution

Therfore, the section lift and drag coefficients can be readily obtained:

$$C_l = \frac{\rho Q_\infty \Gamma(\theta)}{(1/2)\rho Q_\infty^2 c(\theta)} = \frac{4b}{c(\theta)} \sum_{n=1}^{\infty} A_n \sin n\theta \qquad (8.46)$$

$$C_{d_i} = C_l \alpha_i = \frac{4b}{c(\theta)} \sum_{n=1}^{\infty} A_n \sin n\theta \left(\sum_{k=1}^{\infty} k A_k \frac{\sin k\theta}{\sin \theta} \right) \qquad (8.47)$$

The wing aerodynamic coefficients are obtained by the spanwise integration of section coefficients

$$C_L = \int_{-b/2}^{b/2} \frac{C_l(y)c(y)\,dy}{S} = \frac{4b}{S} \int_0^{\pi} \sum_{n=1}^{\infty} A_n \sin n\theta \frac{b}{2} \sin\theta \, d\theta \qquad (8.48)$$

$$C_{D_i} = \int_{-b/2}^{b/2} \frac{C_{d_i}(y)c(y)\,dy}{S} = \frac{2b^2}{S} \int_0^{\pi} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} k A_k A_n \sin k\theta \sin n\theta \, d\theta \qquad (8.49)$$

$$\int_0^{\pi} \sin n\theta \sin k\theta \, d\theta = \begin{pmatrix} 0 & \text{for} & n \neq k \\ \pi/2 & \text{for} & n = k \end{pmatrix}$$

$$C_L = \frac{\pi b^2 A_1}{S} = \pi AR A_1 \qquad (8.51)$$

only the first term will appear

&

$$C_{D_i} = \frac{\pi b^2}{S} \sum_{n=1}^{\infty} n A_n^2 = \pi AR \sum_{n=1}^{\infty} n A_n^2 \qquad (8.52)$$

only the terms where n = k will be left

8.1 Finite Wing: General Spanwise Circulation Distribution

$$C_L = \frac{\pi b^2 A_1}{S} = \pi AR A_1$$

$$C_{D_i} = \frac{\pi b^2}{S} \sum_{n=1}^{\infty} n A_n^2 = \pi AR \sum_{n=1}^{\infty} n A_n^2$$

$$C_{D_i} = \frac{\pi^2 AR^2 A_1^2}{\pi AR} \left[1 + \sum_{n=2}^{\infty} \frac{n A_n^2}{A_1^2} \right] = \frac{C_L^2}{\pi AR} \left[1 + \sum_{n=2}^{\infty} \frac{n A_n^2}{A_1^2} \right] = (C_{D_i})_{\text{elliptic}}(1 + \delta_1) \qquad (8.53)$$

δ_1 includes the higher order terms for n = 2, 3, . . . only the odd terms are considered for a symmetric load distribution

For a given wing aspect ratio,the elliptic wing will have the lowest drag coefficient since for a generic wing planform $\delta_1 \geq 0$ and for the elliptic wing $\delta_1 = 0$.

8.1 Finite Wing: General Spanwise Circulation Distribution

Similarly, the lift coefficient for the general spanwise loading can be formulated as:

$$C_L = \pi AR A_1 = m\left(\alpha - \alpha_{L_0}\right) \qquad (8.54)$$

Assume that the wing is untwisted & therefore $\alpha - \alpha_{L0}$ = const. We define an equivalen 2D wing that has the same lift coefficient C_L .

$$C_L = 2\pi\left(\alpha^* - \alpha_{L_0}\right) \qquad (8.55)$$

The difference between these two cases is due to the wake-induced angle of attack

$$\left(\alpha - \alpha_{L_0}\right) - \left(\alpha^* - \alpha_{L_0}\right) = C_L\left[\frac{1}{m} - \frac{1}{2\pi}\right] \equiv \frac{C_L}{\pi AR}(1+\delta_2) \qquad (8.56)$$

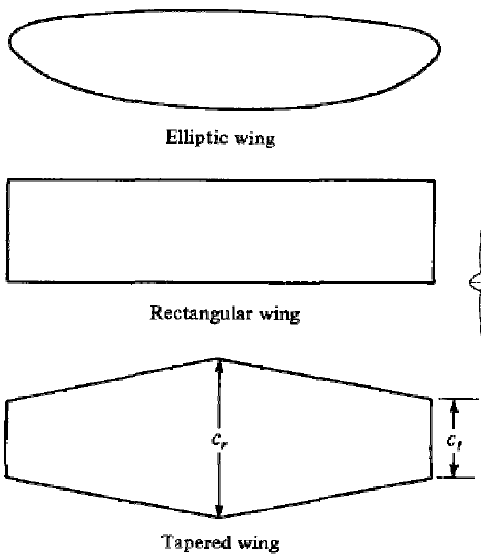
$$\begin{cases} 1+\delta_2 = \pi AR\left[\dfrac{\alpha - \alpha_{L_0}}{\pi AR A_1} - \dfrac{1}{2\pi}\right] \longleftarrow \delta_2 > 0 \\ C_L = \dfrac{\pi b^2 A_1}{S} = \pi AR A_1 \end{cases}$$

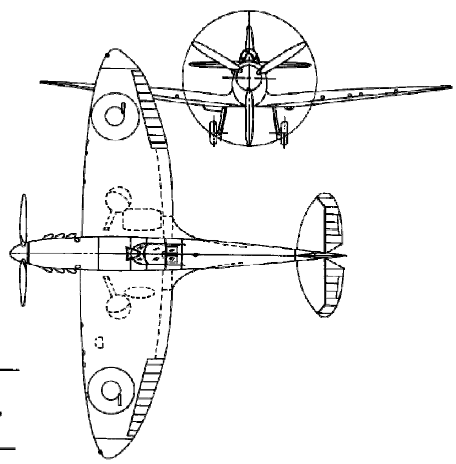
$$C_L = \frac{2\pi(\alpha - \alpha_{L0})}{1 + (2/AR)(1+\delta_2)} \qquad (8.57)$$

Thus, for the elliptic wing $\delta_2 = 0$ and also its lift coefficient is higher than for wings with other spanwise load distributions

8.1 Finite Wing: Varius Planforms

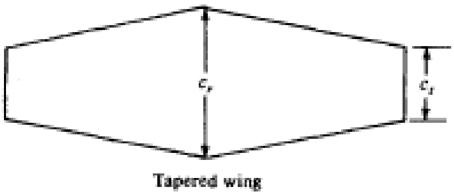
However, elliptic planforms are more expensive to manufacture than, say, a simple rectangular wing. On the other hand, a rectangular wing generates a lift distribution far from optimum. The tapered wing can be designed with a taper ratio, that is, tip chord/root chord = c_t/c_r such that the lift distribution closely approximates the elliptic case.

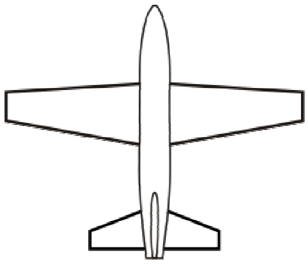


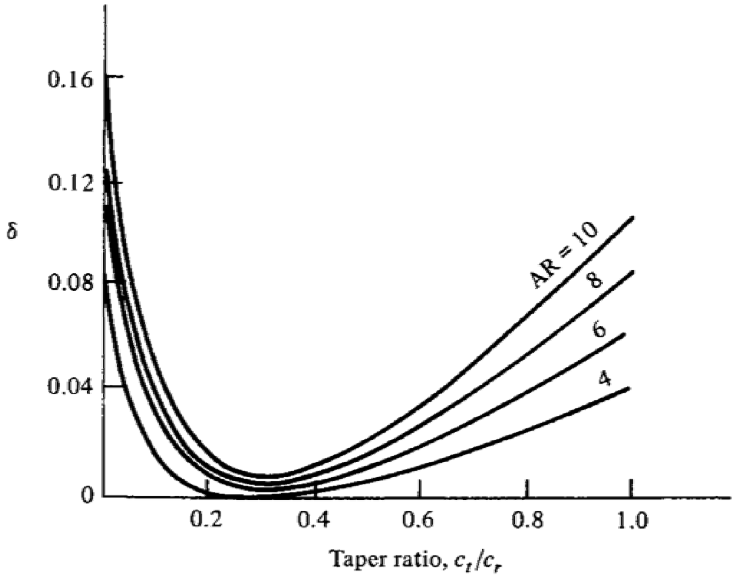


Spitfire airplane

8.1 Finite Wing: Varius Planforms





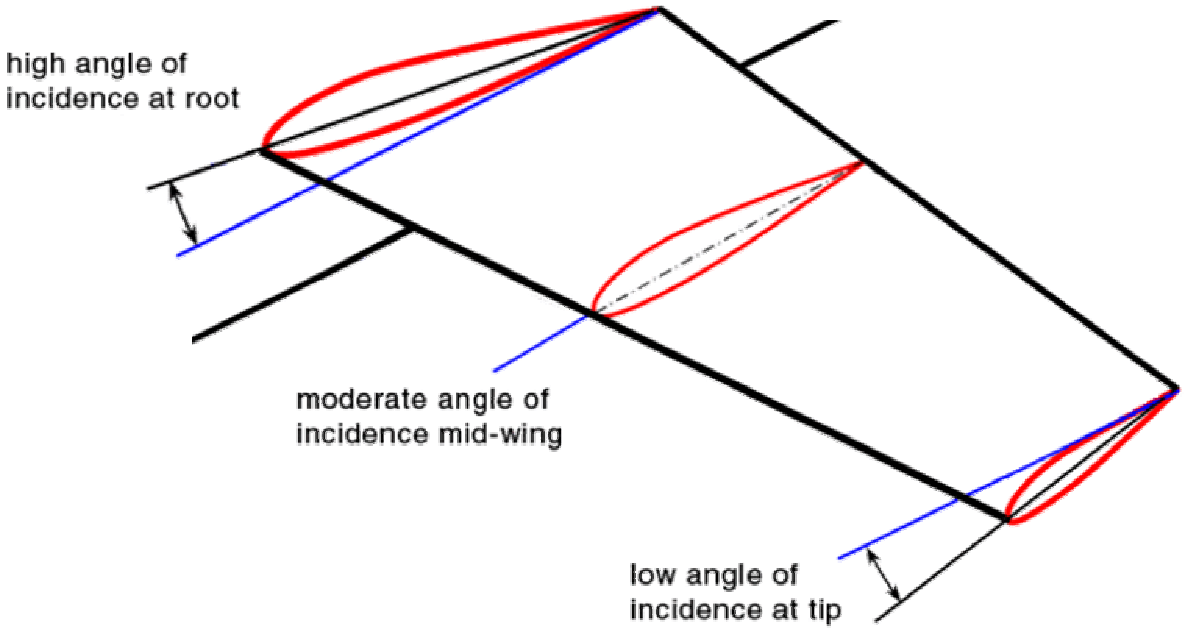


Induced drag factor δ as a function of taper ratio

8.1 Finite Wing: Varius Planforms

8.1 Finite Wing: Twisted Elliptic Wing

The spanwise loading of wings can be varied by introducing twist to the wing planform.



Geometric Twist: Spanwise variation of AOA
Aerodynamic Twist: Spanwise variation of the zero-lift angle (different airfoils)

8.1 Finite Wing: Twisted Elliptic Wing

The governing equation for the coefficients for the circulation distribution (A_n) for the general case that is described using lifting-line theory Eq.(8.44a):

$$\frac{-4b}{m_0(\theta)c(\theta)}\sum_{n=1}^{\infty} A_n \sin n\theta - \sum_{n=1}^{\infty} nA_n \frac{\sin n\theta}{\sin\theta} + \alpha(\theta) - \alpha_{L0}(\theta) = 0 \qquad (8.44a)$$

$$\sum_{n=1}^{\infty} A_n \sin n\theta \left[\frac{4b}{m_0(\theta)c(\theta)} + \frac{n}{\sin\theta}\right] = \alpha(\theta) - \alpha_{L0}(\theta) \qquad (8.58)$$

The closed-form solution for a wing with an elliptic planform & arbitrary twist founded by Filotas.
Consider an elliptic chord distribution as given in Eq. (8.31)

$$c = c_0 \sin\theta \qquad (8.59)$$

Eq. (8.58) $\xrightarrow[AR = 4b/\pi c_0]{m_0 = 2\pi}$ $\displaystyle\sum_{n=1}^{\infty} A_n \sin n\theta \left[\frac{AR}{2} + n\right] = [\alpha(\theta) - \alpha_{L0}(\theta)]\sin\theta$

a Fourier series representation coefficients are given by

$$A_n = \frac{2}{\pi}\frac{1}{AR/2 + n}\int_0^{\pi} [\alpha(\theta) - \alpha_{L0}(\theta)] \sin\theta \sin n\theta \, d\theta \qquad (8.60)$$

8.1 Finite Wing: Twisted Elliptic Wing

To find the wing lift coefficient:

$$A_1 = \frac{2}{\pi}\frac{1}{\text{Æ}/2+1}\int_0^\pi [\alpha(\theta)-\alpha_{L0}(\theta)]\sin^2\theta\, d\theta$$
$$= \frac{4}{\pi}\frac{1}{\text{Æ}+2}\int_0^\pi [\alpha(\theta)-\alpha_{L0}(\theta)]\sin^2\theta\, d\theta$$

$$C_L = \frac{\pi b^2 A_1}{S} = \pi \text{Æ} A_1$$

$$C_L = \frac{4\text{Æ}}{\text{Æ}+2}\int_0^\pi [\alpha(\theta)-\alpha_{L0}(\theta)]\sin^2\theta\, d\theta \qquad (8.61)$$

Example: Consider a wing with a linear twist

$$\alpha(y) = \alpha \pm \alpha_0\left|\frac{y}{b/2}\right| = \alpha \pm \alpha_0|\cos\theta|$$

The effect of the twist can be analyzed by taking the variable part of α(y) only, and adding the contribution of the constant angle of attack later.

$$\alpha(y) = \alpha_0|\cos\theta|$$

8.1 Finite Wing: Twisted Elliptic Wing

Using Eq. (8.60) to compute the coefficients A_n

$$\frac{A_n}{\alpha_0} = \frac{4}{\pi}\frac{1}{\text{Æ}/2+n}\int_0^{\pi/2}\cos\theta\sin\theta\sin n\theta\, d\theta = \frac{2}{\pi}\frac{1}{\text{Æ}/2+n}\int_0^{\pi/2}\sin 2\theta\sin n\theta\, d\theta$$
$$= \frac{2}{\pi}\frac{1}{\text{Æ}/2+n}\left[\frac{\sin(n-2)\theta}{2(n-2)} - \frac{\sin(n+2)\theta}{2(n+2)}\right]_0^{\pi/2}$$
$$= \frac{1}{\pi}\frac{1}{\text{Æ}/2+n}\left[\frac{\sin(n-2)\pi/2}{(n-2)} - \frac{\sin(n+2)\pi/2}{(n+2)}\right]$$

Evaluating the individual Coefficients for a wing with

$$\text{Æ} = 6$$
$$\alpha(y) = \alpha_0|\cos\theta|$$

substituting into Eq. (8.42)

$$\Gamma(\theta) = 2bQ_\infty\sum_{n=1}^{\infty} A_n \sin n\theta$$

$$\Gamma(\theta) = \frac{2bQ_\infty\alpha_0}{\pi}\left[\frac{1}{3}\sin\theta + \frac{1}{5}\sin 3\theta - \frac{1}{42}\sin 5\theta + \frac{2}{225}\sin 7\theta - \cdots\right] \leftarrow \alpha = \alpha_0|\cos\theta|$$

$$\Gamma(\theta) = \frac{2bQ_\infty\alpha_0}{\pi}\left[-\frac{1}{3}\sin\theta - \frac{1}{5}\sin 3\theta + \frac{1}{42}\sin 5\theta - \frac{2}{225}\sin 7\theta - \cdots\right] \leftarrow \alpha = -\alpha_0|\cos\theta|$$

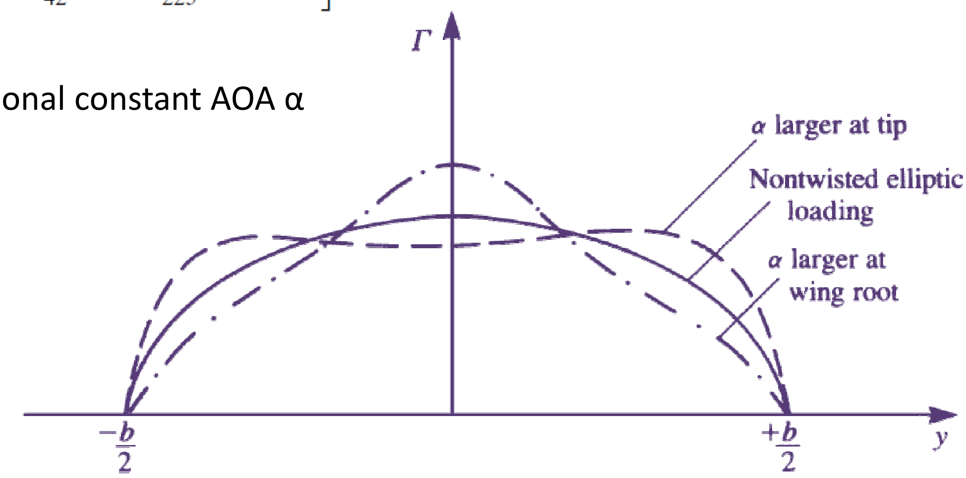
8.1 Finite Wing: Twisted Elliptic Wing

$$\Gamma(\theta) = \frac{2bQ_\infty\alpha_0}{\pi}\left[\frac{1}{3}\sin\theta + \frac{1}{5}\sin 3\theta - \frac{1}{42}\sin 5\theta + \frac{2}{225}\sin 7\theta - \cdots\right]$$

$$\Gamma(\theta) = \Gamma_{\max}[1-\cos^2\theta]^{1/2} = \Gamma_{\max}\sin\theta$$

$$\Gamma(\theta) = \frac{2bQ_\infty\alpha_0}{\pi}\left[-\frac{1}{3}\sin\theta - \frac{1}{5}\sin 3\theta + \frac{1}{42}\sin 5\theta - \frac{2}{225}\sin 7\theta - \cdots\right]$$

Combining with an additional constant AOA α



- A larger AOA at the tip will increase the load there.
- A larger AOA near the wing root will increase the loading there.

8.1 Finite Wing: Conclusions from Lifting-Line Theory

The most important result of the lifting-line theory is its ability to establish the effect of wing aspect ratio on the lift slope and induced drag. Some of the more important conclusions are:

1. The wing lift slope $dC_L/d\alpha$ decreases as wing aspect ratio becomes smaller.

2. The induced drag of a wing increases as wing aspect ratio decreases.

3. A wing with elliptic loading will have the lowest induced drag and the highest lift, as indicated by Eqs. (8.53) and (8.57).

4. This theory also provides valuable information about the wing's spanwise loading and about the existence of the trailing vortex wake.

5. The theory is limited to small disturbances and large aspect ratio and, for example, Eq. (8.6), which requires that the wake be aligned with the local velocity, was not addressed at all (because of the small angle of attack assumption).

6. Using the results of this theory we must remember that the drag of a wing includes the induced drag portion (predicted by this model) plus the viscous drag, which must be taken into account.